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# Solutions of the Space-Time Fractional Foam Drainage Equation and the Fractional Klein-Gordon Equation by Use of Modified Kudryashov Method

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**Abstract-** In this work, we handle the space-time fractional foam drainage equation and the space-time fractional Klein Gordon equation to solve analytically. Firstly, we use fractional traveling wave transformations to convert fractional nonlinear partial differential equations to nonlinear ordinary differential equations. Next, the modified Kudryashov method is applied to find exact solutions of these equations.

**Index Terms- -** the space-time fractional foam drainage equation; space-time fractional Klein Gordon equation; the modified Kudryashov method .

## 1. INTRODUCTION

Fractional order differential equations, are the generalized type of the classical differential equations of integer order, plays an outstanding role from the viewpoint of applications in chemistry, physics and engineering. In the past centuries, the fractional derivatives and integrals were thought to be the subject of the theoretical mathematics. However, in the few decades, many studies have implied that the fractional phenomena is related with not only the pure mathematics, but also the applied sciences such as fluid mechanics, arterial mechanics, optical fibers, geochemistry, plasma physics and so on. Due to the neglect of the effects in classical integer order models, the advantage of the fractional derivatives become apparent in modelling mechanical and electrical properties of real materials. The fractional derivatives construct a basis to describe the features of systems of mathematical modelling and simulation of systems that leads to nonlinear fractional differential equations (FDEs) and to solve such equations [Podlubny (1999)]. In recent years, numerous effective methods for solving these system have been found in the most useful works on nonlinear FDEs. Such as, the sub-equation method [2,5,6,14], the exp-function method [16], the first integral method [11,15], the complex transform method [12] and so on. The common of these methods is based on the homogenous balance principle. In this study, firstly we will describe the modified Kudryashov method which is proposed by N. A. Kudryashov [10] and applied in many studies to construct the exact analytical solutions of nonlinear differential equations. This method is also based on the homogenous balance principle. Therefore, it can be applied to solve the fractional order nonlinear equations. Then, we will apply the proposed method to the space-time fractional foam drainage equation and the fractional Klein-Gordon equation by the help of Jumarie's modified Riemann- Liouville derivative.

## 2. PRELIMINARIES AND THE MODIFIED KUDRYASHOV METHOD

Jumarie's modified Riemann- Liouville derivative is defined as [7,8]:

$$\mathrm{D}_x^\alpha f(x)=\begin{cases}\dfrac{1}{\Gamma(-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x(x-\eta)^{-\alpha-1}\left[f(\eta)-f(0)\right]d\eta, & \alpha<0,\\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x(x-\eta)^{-\alpha}\left[f(\eta)-f(0)\right]d\eta, & 0<\alpha<1,\\ (f^{(n)}(x))^{\alpha-n}, \quad n\le\alpha<n+1, & n\ge1.\end{cases} \tag{1}$$

where

$$D_x^\alpha f(x):=\lim_{h\downarrow0}h^{-\alpha}\sum_{k=0}^{\infty}(-1)^k f[x+(\alpha-k)h]. \tag{2}$$

In addition, some properties for the proposed modified Riemann-Liouville derivative are given in [7,8] as follows:

$$D_t^\alpha t^\gamma=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}t^{\gamma-\alpha}, \qquad \gamma>0, \tag{3}$$

$$D_t^\alpha(f(t)g(t))=g(t)D_t^\alpha f(t)+f(t)D_t^\alpha g(t), \tag{4}$$

$$D_t^\alpha f[g(t)]=f_g^{'}[g(t)]D_t^\alpha g(t)=D_g^\alpha f[g(t)](g'(t))^\alpha, \tag{5}$$

which are the direct consequence of

$$D^\alpha x(t)=\Gamma(1+\alpha)Dx(t). \tag{6}$$

We present the main steps of the modified Kudryashov method as follows [1,3,4,9,10]:

For a given nonlinear FDEs for a function $u$ of independent variables, $X=(x,y,z,...,t)$:

$$F(u,u_t,u_x,u_y,u_z,...,D_t^\alpha u,D_x^\alpha u,D_y^\alpha u,D_z^\alpha u,...)=0 \tag{7}$$

where $D_t^{\alpha}u$, $D_x^{\alpha}u$, $D_y^{\alpha}u$ and $D_z^{\alpha}u$ are the modified Riemann-Liouville derivatives of *u* with respect to *t, x, y* and *z*. *F* is a polynomial in *u = u(x,y, z,..., t)* and its various partial derivatives, in which the nonlinear terms and highest order derivatives are involved.

**Step 1:** We investigate the traveling wave solutions of Eq.(7) of the form:

$$u(x, y, z, ..., t) = u(\eta),$$

$$\eta = \frac{\omega x^{\beta}}{\Gamma(1+\beta)} + \frac{\varepsilon y^{\gamma}}{\Gamma(1+\gamma)} + \frac{\sigma z^{\delta}}{\Gamma(1+\delta)} + \cdots + \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}, \quad (8)$$

where $\omega$, $\varepsilon$, $\sigma$ and $\lambda$ are arbitrary constants. Then Eq.(7) reduces to a nonlinear ordinary differential equation of the form:

$$G(u, u_{\eta}, u_{\eta\eta}, u_{\eta\eta\eta}, \ldots) = 0. \quad (9)$$

**Step 2:** We suppose that the exact solutions of Eq.(9) can be obtained in the following form:

$$u(\eta) = \sum_{i=0}^{M} a_i Q^i(\eta) \quad (10)$$

where $Q = \frac{1}{1 \pm e^{\eta}}$ and the function *Q* is the solution of equation $Q_{\eta} = Q^2 - Q.$

**Step 3:** According to the method, we assume that the solution of Eq.(9) can be expressed in the form

$$u(\eta) = a_N Q^N + \cdots. \quad (11)$$

Calculation of value *N* in formula (11) that is the pole order for the general solution of Eq.(9).

In order to determine the value of *N* we balance the highest order nonlinear terms in Eq.(9) analogously as in the classical Kudryashov method. Supposing $u^l(\eta)u^{(s)}(\eta)$ and $(u^{(p)}(\eta))^r$ are the highest order nonlinear terms of Eq.(9) and balancing the highest order nonlinear terms we have:

$$N = \frac{s - rp}{r - l - 1}. \quad (12)$$

**Step 4:** Substituting Eq.(10) into Eq.(9) and equating the coeffcients of $Q^i$ to zero, we get a system of algebraic equations. By solving this system, we obtain the exact solutions of Eq.(9).

## 3. APPLICATIONS

### 3.1. *The Space-Time Fractional Foam Drainage Equation*

We first apply the method to the space-time fractional foam drainage equation in the form:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{2} u \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + 2u^2 \frac{\partial^{\beta} u}{\partial x^{\beta}} + \left(\frac{\partial^{\beta} u}{\partial x^{\beta}}\right)^2 \quad (13)$$

where $0 < \alpha, \beta \le 1$, $x > 0$ and $u$ is the function of *(x, t)*.

This equation describes the evolution of the vertical density profile of a foam under gravity. Their uses extended from packaging, car manufacturing to ore-separation and brewing [Cox *et al.* (2002)]. By considering the traveling wave transformation:

$$u(x,t) = u(\eta),$$

$$\eta = \frac{\omega x^{\beta}}{\Gamma(1+\beta)} - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} \quad (14)$$

where $\omega$ and $\lambda$ are constants. Then Eq.(13) can be reduced to the following ordinary differential equation:

$$-\lambda u' + \frac{1}{2}\omega^2 u u'' + 2\omega u^2 u' + \omega^2 u'^2 = 0. \quad (15)$$

Also we take

$$u(\eta) = a_0 + a_1 Q + \cdots + a_N Q^N \quad (16)$$

where $Q = \frac{1}{1 \pm e^{\eta}}$. We note that the function *Q* is the solution of $Q_{\eta} = Q^2 - Q.$ Balancing the the linear term of the highest order with the highest order nonlinear term in Eq.(15), we compute

$$N = 1. \quad (17)$$

Thus, we have

$$u(\eta) = a_0 + a_1(\eta) \quad (18)$$

and taking the derivatives of $u(\eta)$ with respect to $\eta$, we obtain

$$u_{\eta} = a_1 Q^2 - a_1 Q, \quad (19)$$

$$u_{\eta\eta} = 2a_1 Q^3 - 3a_1 Q^2 + a_1 Q. \quad (20)$$

Substituting Eq.(19) and Eq.(20) into Eq.(15) and collecting the coefficient of each power of $Q^i$ setting each of coefficient to zero, solving the resulting system of algebraic equations we obtain the following solutions:

$$a_0 = \frac{\omega}{2}, \quad a_1 = -\omega, \quad \lambda = \frac{\omega^3}{4}. \quad (21)$$

Inserting Eq.(15) into Eq.(21), we obtain the following solutions of Eq.(13)

$$u_1(x,t) = \omega\left(\frac{1}{2} - \frac{1}{1 + e^{\frac{\omega x^{\beta}}{\Gamma(1+\beta)} - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}}}\right), \quad (22)$$

$$u_2(x,t) = \omega\left(\frac{1}{2} - \frac{1}{1 - e^{\frac{\omega x^{\beta}}{\Gamma(1+\beta)} - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}}}\right). \quad (23)$$

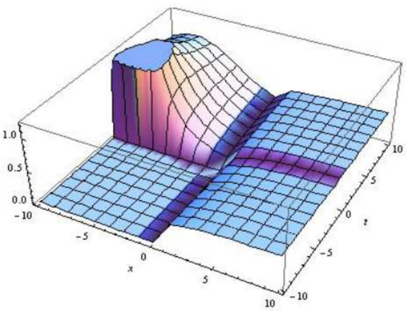

Fig.1. The graph of $u_1(x,t)$ for $\alpha = 0.5$, $\beta = 0.5$ and $\omega = 1$.

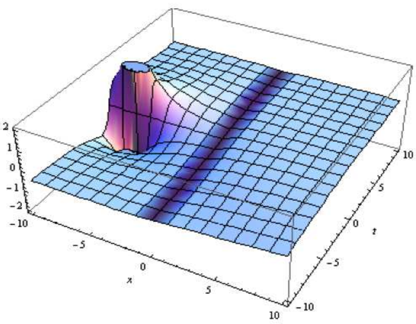

Fig.2. The graph of $u_1(x,t)$ for $\alpha = 1$, $\beta = 0.5$ and $\omega = 1$.

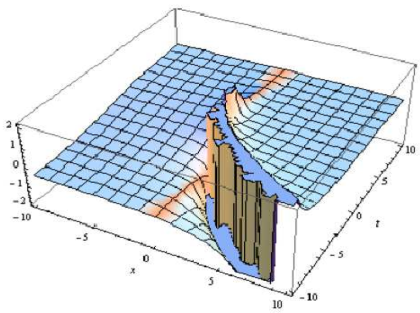

Fig.3. The graph of $u_2(x,t)$ for $\alpha = 1$, $\beta = 0.5$ and $\omega = 1$.

### 3.2. *The Fractional Klein-Gordon Equation*

We, next consider the fractional Klein-Gordon equation defines as:

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = \frac{\partial^2 u}{\partial x^2} + bu + cu^3 \qquad (24)$$

where $t > 0$, $0 < \alpha \le 1$ and $u$ is the function of $(x, t)$. This equation defines a motion of pseudoscalar field whose quanta are spinless particles. This equation also describes the quantum amplitude for finding a point particle in various places. By considering the traveling wave transformation:

$$u(x,t) = u(\eta), \quad \eta = \omega x - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \qquad (25)$$

where $\omega$ and $\lambda$ are constants. Then Eq.(24) can be reduced to the following ordinary differential equation:

$$\lambda^2 u'' = \omega^2 u'' + bu + cu^3 \qquad (26)$$

Also we take

$$u(\eta) = a_0 + a_1 Q + \cdots + a_N Q^N \qquad (27)$$

where $Q = \frac{1}{1 \pm e^\eta}$. We note that the function $Q$ is the solution of $Q_\eta = Q^2 - Q$. Balancing the the linear term of the highest order with the highest order nonlinear term in Eq.(26), we compute

$$N = 1. \qquad (28)$$

Thus, we have

$$u(\eta) = a_0 + a_1(\eta) \qquad (29)$$

and taking the derivatives of $u(\eta)$ with respect to $\eta$, we obtain

$$u_\eta = a_1 Q^2 - a_1 Q, \qquad (30)$$

$$u_{\eta\eta} = 2a_1 Q^3 - 3a_1 Q^2 + a_1 Q. \qquad (31)$$

Substituting Eq.(30) and Eq.(31) into Eq.(26) and collecting the coefficient of each power of $Q^i$ setting each of coefficient to zero, solving the resulting system of algebraic equations we obtain the following solutions:

**Case1:** $a_0 = \sqrt{\frac{b}{c}}, \quad a_1 = -2\sqrt{\frac{b}{c}}, \quad \lambda = -\sqrt{-2b+\omega^2}$. (32)

$$u_1(x,t) = \sqrt{\frac{b}{c}}\left(1 - 2\frac{1}{1 + e^{\omega x + \frac{\left(\sqrt{-2b+\omega^2}\right)t^\alpha}{\Gamma(1+\alpha)}}}\right), \qquad (33)$$

$$u_2(x,t) = \sqrt{\frac{b}{c}}\left(1 - 2\frac{1}{1 - e^{\omega x + \frac{\left(\sqrt{-2b+\omega^2}\right)t^\alpha}{\Gamma(1+\alpha)}}}\right). \qquad (34)$$

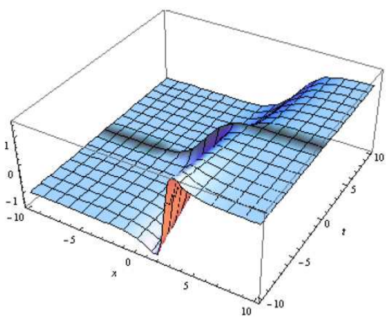

Fig.4. The graph of $u_1(x,t)$ for $\alpha = 0.25$, $b = c = -1$ and $\omega = 1$.

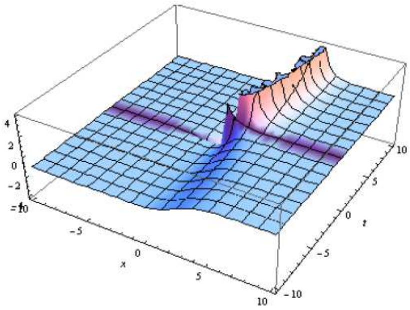

Fig.5. The graph of $u_2(x,t)$ for $\alpha = 0.25,\ \ b = c = -1$ and $\omega = 1$.

**Case2:** $a_0 = \sqrt{\frac{b}{c}},\ \ a_1 = -2\sqrt{\frac{b}{c}},\ \ \lambda = \sqrt{-2b+\omega^2}.$ (35)

$$u_3(x,t) = \sqrt{\frac{b}{c}}\left(1 - 2\frac{1}{1+e^{\omega x - \frac{\left(\sqrt{-2b+\omega^2}\right)t^\alpha}{\Gamma(1+\alpha)}}}\right), \qquad (36)$$

$$u_4(x,t) = \sqrt{\frac{b}{c}}\left(1 - 2\frac{1}{1-e^{\omega x - \frac{\left(\sqrt{-2b+\omega^2}\right)t^\alpha}{\Gamma(1+\alpha)}}}\right). \qquad (37)$$

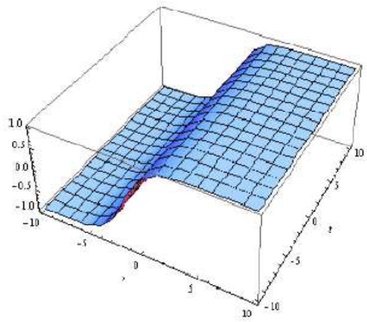

Fig.6. The graph of $u_3(x,t)$ for $\alpha = 1,\ \ b = c = -1$ and $\omega = 1$.

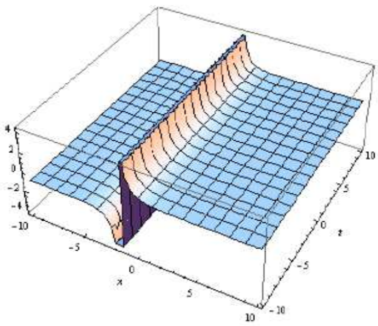

Fig.7. The graph of $u_4(x,t)$ for $\alpha = 1,\ \ b = c = -1$ and $\omega = 1$.

**Case3:** $a_0 = -\sqrt{\frac{b}{c}},\ a_1 = 2\sqrt{\frac{b}{c}},\ \ \lambda = -\sqrt{-2b+\omega^2}.$ (38)

$$u_5(x,t) = \sqrt{\frac{b}{c}}\left(-1 + 2\frac{1}{1+e^{\omega x + \frac{\left(\sqrt{-2b+\omega^2}\right)t^\alpha}{\Gamma(1+\alpha)}}}\right), \qquad (39)$$

$$u_6(x,t) = \sqrt{\frac{b}{c}}\left(-1 + 2\frac{1}{1-e^{\omega x + \frac{\left(\sqrt{-2b+\omega^2}\right)t^\alpha}{\Gamma(1+\alpha)}}}\right). \qquad (40)$$

**Case4:** $a_0 = -\sqrt{\frac{b}{c}},\ a_1 = 2\sqrt{\frac{b}{c}},\ \ \lambda = \sqrt{-2b+\omega^2}.$ (41)

$$u_7(x,t) = \sqrt{\frac{b}{c}}\left(-1 + 2\frac{1}{1+e^{\omega x - \frac{\left(\sqrt{-2b+\omega^2}\right)t^\alpha}{\Gamma(1+\alpha)}}}\right), \qquad (42)$$

$$u_8(x,t) = \sqrt{\frac{b}{c}}\left(-1 + 2\frac{1}{1-e^{\omega x - \frac{\left(\sqrt{-2b+\omega^2}\right)t^\alpha}{\Gamma(1+\alpha)}}}\right). \qquad (43)$$

## 4. CONCLUSION

In this work, we find the analytical solutions of the space-time fractional foam drainage equation and the fractional Klein-Gordon equation by using the modified Kudryashov method. Also, we use Jumarie's modified Riemann-Liouville derivation formulas and properties to reduce the fractional order differential equations into Riccati type equations. It can be seen clearly that the method is suitable for solving Riccati equations since it is also based on the homogenous balance principle. The obtained solutions are rational function solutions whose structures are in the traveling wave form.

This method is effective, useful and easily computable with the help of computer algebra system Mathematica. Therefore, it can be applied to other nonlinear fractional differential equations. According to the balancing degree of the equations, hyperbolic function solutions can also be obtained.

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